MATH 2050 C Lecture 13 (Feb 28)

Last time: "Subsequences"

Let $(x_n)_{n\in\mathbb{N}} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \dots, x_n, \dots)$

Take natural no. $n_1 < n_2 < n_3 < n_4 < n_5 < n_6 < \cdots$ Strictly increasing

any subseq: $(x_{n_k})_{k\in\mathbb{N}} = (x_{n_1}, x_{n_2}, x_{n_3}, \dots, x_{n_k}, \dots \dots)$

Thm: $(x_n) \to x \Rightarrow$ ANY subseq. $(x_{n_k}) \longrightarrow x$

Thm: "(x_n) does NOT converge to x"

$\Leftrightarrow \exists\ \varepsilon_0 > 0$ and a subseq. (x_{n_k}) s.t.

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \forall\, k \in \mathbb{N}.$$

Recall: "MCT": (x_n) bdd & monotone $\Rightarrow$ (x_n) convergent

[E.g.) $(x_n) = ((-1)^n)$ bdd, but NOT monotone, NOT convergent.]

Q: What if (x_n) is ONLY bdd?

Bolzano-Weierstrass Thm: "BWT" "Compactness" (MATH 3070).

(x_n) bdd $\Rightarrow \exists$ subseq. (x_{n_k}) which is convergent.

But not unique!

Example: $(x_n) = ((-1)^n)$ has a convergent subseq.

namely $(x_{2k}) = (1, 1, 1, 1, \dots) \to 1$

another choice $(x_{2k-1}) = (-1, -1, -1, -1, \dots) \to -1$ $\neq$

Proof: We will prove it using "Nested Interval Property" (NIP)

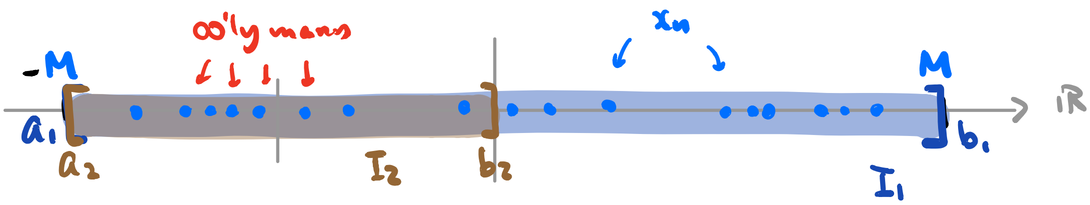
[Recall: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ nested, closed & bdd

$\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$ If furthermore $\lim \text{Length}(I_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.]

Goal: Construct I_n inductively satisfying the hypothesis above.

Given a bdd seq (x_n), by def^{n}, $\exists\, M > 0$ s.t. $|x_n| \le M \ \forall n \in \mathbb{N}$

i.e $\forall n \in \mathbb{N}$, $x_n \in [-M, M] =: I_1 = [a_1, b_1]$



Do "method of bisection":

Consider the midpoint $\frac{a_1+b_1}{2}$, then

Case 1: $[a_1, \frac{a_1+b_1}{2}]$ contains infinitely many terms of (x_n)

$\rightsquigarrow$ choose $I_2 := [a_1, \frac{a_1+b_1}{2}] = [a_2, b_2]$.

Case 2: Otherwise.

$\rightsquigarrow$ choose $I_2 := [\frac{a_1+b_1}{2}, b_1] = [a_2, b_2]$

Repeat the process. take a mid pt. $\frac{a_2+b_2}{2}$, choose $I_3 = [a_3, b_3]$.

Inductively, we obtain a seq of intervals:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots\cdots \quad \text{nested, closed \& bdd}$$

s.t. • each I_n contains infinitely many terms of (x_n)

• $\text{Length}(I_n) = \frac{2M}{2^{n-1}} \to 0$ as $n \to \infty$

By "NIP". $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$

Claim: $\exists$ subseq. $(x_{n_k}) \to \xi$

Pf: Take any $x_{n_1} \in I_1$, then since I_2 contains infinitely many terms of (x_n)

$\leadsto$ we can choose $n_2 > n_1$ s.t. $x_{n_2} \in I_2$

$\leadsto$ keep on doing this, we obtain $n_1 < n_2 < n_3 < \dots$ st

$$x_{n_k} \in I_k = [a_k, b_k] \quad \forall k \in \mathbb{N}.$$

i.e. $$a_k \le x_{n_k} \le b_k. \quad \forall k \in \mathbb{N}.$$

Now. $\bigcap_{n=1}^{\infty} I_n = \{\xi\} \Rightarrow \lim a_k = \lim b_k = \xi$.

By Squeeze Thm, we have $\lim_{k\to\infty} (x_{n_k}) = \xi$.

$\square$

As an application of BWT, we prove:

Prop: Let (x_n) be a bdd sequence.

$(x_n) \to x \iff$ ANY convergent subseq. (x_{n_k}) has $\lim_{k\to\infty} (x_{n_k}) = x$

Proof: "$\Rightarrow$" DONE.

"$\Leftarrow$" Suppose NOT, ie (x_n) does NOT converge to x.

By earlier thm. $\exists\ \varepsilon_0 > 0$ & a subseq (x_{n_k}) s.t.

$$|x_{n_k} - x| \ge \varepsilon_0 \quad \forall k \in \mathbb{N} \quad \text{———} (*)$$

By BWT., $(x_{n_k})_k$ bdd $\Rightarrow \exists$ convergent subseq $(x_{n_{k_\ell}})_\ell$ of $(x_{n_k})_k$
($\because$ (x_n) bdd) which is also a subseq of $(x_n)_n$

By hypothesis, $\lim_{\ell\to\infty} (x_{n_{k_\ell}}) = x$ Contradicting $(*)$.

$\square$

Subsequential limits, limsup & liminf

Q: Given a bdd seq. (x_n), what is

$$\mathcal{L} := \{ \ell \in \mathbb{R} \mid \exists \text{ subseq. } (x_{n_k}) \text{ of } (x_n) \text{ s.t. } \lim_{k\to\infty} (x_{n_k}) = \ell \} \text{ ?}$$

Example: If $\lim (x_n) = x$, then $\mathcal{L} = \{x\}$.

Example: $(x_n) = ((-1)^n)$, then $\mathcal{L} = \{1, -1\}$

Remark: BWT $\Rightarrow$ $\mathcal{L} \neq \phi$.

(x_n) bdd $\Rightarrow \exists M > 0$ s.t. $|x_n| \leq M \quad \forall n \in \mathbb{N}$.

So, any convergent subseq. (x_{n_k}) satisfy

$$-M \leq x_{n_k} \leq M \quad \forall k \in \mathbb{N}.$$

$$\overset{\text{as } k\to\infty}{\Longrightarrow} \quad -M \leq \ell \leq M$$

ie $\overset{\text{BWT}}{\phi \neq} \mathcal{L} \subseteq [-M, M]$ is bdd subset of $\mathbb{R}$.

By completeness of $\mathbb{R}$, $\inf \mathcal{L}$, $\sup \mathcal{L}$ both exist.

Def$^{\underline{n}}$:

$$\limsup (x_n) = \overline{\lim} (x_n) := \sup \mathcal{L}$$

$$\liminf (x_n) = \underline{\lim} (x_n) := \inf \mathcal{L}$$

Thm: Let (x_n) be a bdd seq. Define another seq. (u_m) by

$$u_m := \sup \{ x_n \mid n \geq m \} \quad \text{for each } m = 1, 2, 3, \ldots$$

THEN, (u_m) is a decreasing seq. with

$$\lim_{m\to\infty} (u_m) = \inf \{ u_m \mid m \in \mathbb{N} \} = \overline{\lim} (x_n)$$

Proof: [Recall: $S_1 \subseteq S_2 \Rightarrow \sup S_1 \le \sup S_2$]

$$(x_n) = (x_1, x_2, x_3, x_4, x_5, \dots, x_n, \dots)$$

sup $= u_1$ sup $= u_2$

$\forall m \in \mathbb{N}.\quad \{x_n \mid n \ge m\} \supseteq \{x_n \mid n \ge m+1\}$

take sup. $\qquad u_m \ge u_{m+1}$

So, (u_m) is decreasing, and bdd ($\because (x_n)$ bdd)

By MCT, $\lim_{m\to\infty} (u_m) = \inf\{u_m : m \in \mathbb{N}\}$.

Claim 1: $\overline{\lim}(x_n) \le \lim (u_m)$

Pf: Recall $\overline{\lim}(x_n) = \sup \mathcal{L}$. Let $l \in \mathcal{L}$, then by def^n.

$\exists$ subseq. $(x_{n_k}) \to l$. By def^n of u_m (when $m = n_k$)

$$x_{n_k} \le u_{n_k} := \sup\{x_n \mid n \ge n_k\} \quad \forall k \in \mathbb{N}$$

let $k \to \infty$. $\qquad l \le \lim_{k\to\infty} (u_{n_k}) = \lim_{m\to\infty} (u_m)$

(u_{n_k}) is a subseq of the convergent seq (u_m).

Claim 2: $\overline{\lim}(x_n) \ge \lim (u_m)$

Pf: Want to show: $\lim (u_m) \in \mathcal{L}$

We have to find a subseq (x_{n_k}) of (x_n) st

$$(x_{n_k}) \to \lim (u_m)$$

- Choose $n_1 \ge 1$ s.t. $u_1 - 1 < x_{n_1} \le u_1 := \sup\{x_n \mid n \ge 1\}$
- Choose $n_2 > n_1$ s.t. $u_{n_1+1} - \frac{1}{2} < x_{n_2} \le u_{n_1+1} := \sup\{x_n \mid n \ge n_1+1\}$

Do it inductively, we can choose $n_1 < n_2 < n_3 < \cdots$

s.t. $$u_{n_k+1} - \frac{1}{k+1} < x_{n_{k+1}} \leq u_{n_k+1} \quad \forall k \in \mathbb{N}$$

Take $k \to \infty$ above, by Squeeze Thm.

$$\ell = \lim (x_{n_k}) = \lim (u_m) \in \mathcal{L}$$

$\blacksquare$

Remarks

(i) $\overline{\lim}(x_n)$, $\underline{\lim}(x_n) \in \mathcal{L}$

(ii) $\overline{\lim}(x_n)$, $\underline{\lim}(x_n)$ always exist [BUT not $\lim (x_n)$]

provided that (x_n) is bdd.

(iii) $\overline{\lim}(x_n + y_n) \leq \overline{\lim}(x_n) + \overline{\lim}(y_n)$ Pf: Exercise!

(c.f. Limit thms)